Matter-induced vertices for photon splitting in a weakly magnetized plasma

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We evaluate the three-photon vertex functions at order B and B^2 in a weak constant magnetic field at finite temperature and density with on shell external lines. Their application to the study of the photon splitting process leads to consider high energy photons whose dispersion relations are not changed significantly by the plasma effects. The absorption coefficient is computed and compared with the perturbative vacuum result. For the values of temperature and density of some astrophysical objects with a weak magnetic field, the matter effects are negligible.

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I. INTRODUCTION

The photon splitting process in the presence of an external magnetic field was thoroughly studied in the vacuum by Adler [1]. Since then, it has been studied several times over the last years using different techniques and for different regimes [2–5] due to its possible relevance in astrophysical applications (see e.g. [6]).

In the vacuum, the Furry theorem forbids the processes with an odd number of photon vertices. The lowest order contribution comes from the box graph with an insertion of the external magnetic field. In the collinear aproximation (which is always considered in the rest of this paper), this matrix element vanishes due to Lorentz and gauge invariance. Thus, the first non-zero contribution to the amplitude comes from the hexagon which is of order B^3. Adler also studied how the propagation of photons is modified by the presence of a magnetic field. He found that there are two linearly polarized modes and due to CP invariance, all processes with an odd number of perpendicularly polarized photons vanish. This along with the induced birefringence in the polarized vacuum enforces that the only allowed channel is $\| \to \perp\perp$. The absorption coefficient is

$$\kappa = \frac{\alpha^3}{15\pi^2}\left(\frac{13}{315}\right)^2\left(\frac{\omega}{m}\right)^5\left(\frac{B\sin\theta}{B_{cr}}\right)^6 m, \tag{1}$$

where θ is the angle between the magnetic field and the direction of the incident photon, ω is the energy of the incident photon and $|e|B_{cr} = m^2 \simeq 4.4\ 10^9\ T$.

In the presence of a hot or dense medium, explicit Lorentz invariance is lost since now there is a privileged reference frame, namely the frame in which the system is at rest. Therefore the box diagram does not need to vanish. In addition, for a non zero net electron density, the ground state is not C invariant and hence the amplitudes with an odd number of photon vertices do not need to vanish either. Since the environment in astrophysical objects is not exactly the vacuum, it is worth studying the effects that density and temperature can have in the photon splitting process.

As far as we know, a quantitative analysis of the matter effects on the photon splitting has been reported in a few papers [7–9]. In Ref. [9], the absorption coefficient is computed by including the plasma effects on the dispersion relations but using the vacuum vertices from the Euler-Heisenberg effective Lagrangian. In Ref. [7], a one-loop thermal effective Lagrangian for a constant electromagnetic field is computed and from that the relevant matrix element of order B is read by following the same procedure that is used in the absence of a medium [1,10]. In Ref. [8], the same approach is used starting from a two-loop effective Lagrangian. Thus, the amplitudes reported in Refs. [7,8] have an analytic dependence on the frequencies of the photons in the process. While this approach works properly in the vacuum, it can have a flaw when it is applied for the study of the plasma effects.

In the vacuum, a consequence of Lorentz and gauge invariance is that the effective action for the electromagnetic field displays an analytic dependence on the field strength and its derivatives. It is just this analytic behaviour which enables a well defined derivative expansion of the effective action whose first few terms of the Taylor's series can be computed from the Euler-Heisenberg Lagrangian [1]. Moreover, because of Adler's theorem, the three-point photon vertex in the collinear approximation computed from the Euler-Heisenberg Lagrangian is exact to order B^3.

On the other hand, in a medium, Lorentz invariance is lost and the effective action displays a non-analytic behaviour [11,12]. This invalidates the identification of proper photon vertices based on an effective Lagrangian computed by a derivative expansion technique. In order to clarify this essential point, let us consider the thermal two-point polarization tensor in QED at high temperature. In the static limit, the only non-zero component of this tensor is $\Pi^{00}(\omega = 0, \mathbf{k}) = -m_D^2 + \mathcal{O}(k^2)$, where $m_D^2 = e^2T^2/3$ is the Debye mass of static screening. Then, the corresponding lowest order terms of a derivative expansion in

[1]When the masses are different from zero and $n \geq 3$, the one-particle irreducible n-point functions are analytic functions of all its arguments, $p_i^2 = \omega_i^2 - \mathbf{p}_i^2$ and $p_i \cdot p_j = \omega_i\omega_j - \mathbf{p}_i \cdot \mathbf{p}_j$ for $(p_1, \ldots, p_{n-1}) = (0, \ldots, 0)$, so that its Taylor series can be guessed from the behaviour for $\omega_i = 0$.

the effective action are $\Gamma \sim \int d^4x \left(\frac{1}{2}m_D^2 A_0^2 + \mathcal{O}(\mathbf{E}^2) + \mathcal{O}(\mathbf{B}^2)\right)$. Here $A_0, \mathbf{E}$ and $\mathbf{B}$ represent the temporal component of the gauge field, the electric and the magnetic field respectively. However, when $\omega \approx k$, the non-zero components are $\Pi^{ij}(\omega = k, \mathbf{k}) = m_t^2(\delta^{ij} - \hat{k}^i\hat{k}^j)$, where $m_t^2 = e^2T^2/6$ is the transverse photon mass defined below in Eq. (3). Now, the corresponding effective action is

$$\Gamma \sim \int d^4x \, \frac{m_t^2}{2} \left(\mathbf{A}^2 + \mathrm{div}\mathbf{A} \, \frac{1}{\nabla^2} \, \mathrm{div}\mathbf{A}\right), \tag{2}$$

which has a non-local dependence in the fields but is gauge invariant. This differs from the corresponding effective action based on the derivative expansion and would not have been obtained from an expansion about a constant field. Certainly, in order to describe a dynamical process involving propagation of photons with a dispersion relation $\omega^2 \approx k^2$, the relevant matrix elements should be computed from an effective action similar to the one in the Eq. (2).

In this paper, we will directly compute the three point amplitude using the Schwinger propagator for an electron in an external magnetic field when the collinear external momenta are on shell. Expanding the three-photon vertex at lowest order in the external field and retaining the relevant pieces we will obtain the box contribution. Proceeding in the same way, we will also obtain the order B^2 contribution which would come from the pentagon graph. This leads to Eqs. (25) and (33) below, which are the main findings of this paper. Both of them display a non-analytic dependence on the photon energies. By computing several components of the amplitudes, we have verified that the Ward identities are satisfied.

The presence of a medium not only affects the amplitudes, allowing some of them not to vanish, but also modifies the photon propagation properties. However, at high energy we expect this modification to be negligible compared with that due to the magnetic field. In order to estimate under which conditions this is true, we recall that the dispersion relation of high energy photons in the absence of an external field is $\omega \simeq \sqrt{p^2 + m_t^2}$, where m_t is known as the transverse mass of the photon [13]

$$m_t^2 = \frac{4\alpha}{\pi}\int_0^\infty dp\,\frac{p^2}{\varepsilon_p}\left(n_F(\varepsilon_p-\mu)+n_F(\varepsilon_p+\mu)\right), \tag{3}$$

with $n_F(x) = 1/(e^{\beta x}+1)$ and $\varepsilon_p = \sqrt{p^2+m^2}$ the energy of an electron of momentum p. The transverse mass is of the same order as the plasma frequency. The refractive index is found to be $n \approx 1 - \frac{m_t^2}{2p^2}$. On the other hand, the refractive indices due to vacuum birefringence are [1]

$$n_\parallel^\perp = 1 + \alpha a_\parallel^\perp \left(\frac{B\sin\theta}{B_{cr}}\right)^2, \quad a_\parallel = \frac{4}{90\pi}, \quad a^\perp = \frac{7}{90\pi}. \tag{4}$$

In this paper we follow the convention of Adler [1], where $\perp$ ($\|$) means the magnetic field of the photon is perpendicular (parallel) to the plane containing the external magnetic field and the photon's momentum vector. Thus, we must require

$$p \gtrsim \frac{1}{\sqrt{2\alpha a}}\left(\frac{B_{cr}}{B\sin\theta}\right) m_t \quad \longrightarrow \quad p \gg 10^2\left(\frac{B_{cr}}{B}\right) m_t \equiv \omega_0. \tag{5}$$

This inequality specifies the condition that is necessary in order to neglect the medium effects on the photon propagation. We will find that in these conditions the contribution of plasma effects to the absorption coefficient is smaller than the vacuum contribution.

In the next section we describe the computation of the box and pentagon amplitudes. The following section is devoted to obtaining the absorption coefficient and evaluating it in several situations. Finally in section IV we give a brief summary.

II. THE BOX AND PENTAGON AMPLITUDES

In order to compute the three photon amplitude we need the fermion propagator in the presence of an external magnetic field. It was computed by Schwinger a long time ago [14] using the proper time method

$$\begin{aligned} G(x,x') = &-i(i\partial\!\!\!/ - e A\!\!\!/ + m)\int_{-\infty}^0 ds\,\frac{-i}{(4\pi)^2 s^2}\, e^{i(m^2-i\epsilon)s} e^{\Phi(x,x')}\, e^{i\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}s} \\ &\times \exp\left(-\frac{1}{2}\mathrm{Tr}\ln[(eFs)^{-1}\sinh(eFs)] + \frac{i}{4}(x-x')eF\coth(eFs)(x-x')\right), \end{aligned} \tag{6}$$

with

$$\Phi(x,x')\equiv -ie\int_{x'}^{x} d\tilde{x}^\mu (A_\mu(\tilde{x}) + \frac{1}{2}F_{\mu\nu}(\tilde{x}-x')^\nu). \tag{7}$$

It can be expressed as

$$G(x,x') = e^{\Phi(x,x')}\,\tilde{G}(x-x'), \tag{8}$$

where $\tilde{G}(x-x')$ is explicitly invariant under translations while the phase factor is not. We need the expansion of $\tilde{G}$ in powers of the magnetic field, for which we obtain in Fourier space

$$\begin{aligned}
\tilde{G}(p) &= \tilde{G}_0(p) + \tilde{G}_1(p) + \tilde{G}_2(p) + \mathcal{O}(B^3), \\
\tilde{G}_0(p) &= \frac{m\mathbf{1}+\not{p}}{p^2-m^2}, \\
\tilde{G}_1(p) &= \frac{e}{(p^2-m^2)^2}(\tilde{F}_{\mu\nu}\gamma^\mu\gamma^5 p^\nu + \frac{m}{2}F_{\mu\nu}\sigma^{\mu\nu}), \\
\tilde{G}_2(p) &= -2e^2\frac{m\mathbf{1}+\not{p}}{(p^2-m^2)^4}(F_{\mu\alpha}F^\alpha_{\ \nu}p^\mu p^\nu) + \frac{2e^2}{(p^2-m^2)^3}(F_{\mu\alpha}F^\alpha_{\ \nu}\gamma^\mu p^\nu),
\end{aligned} \tag{9}$$

with $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$. We have arrived at these results by expanding the integrand of $\tilde{G}$ in powers of B, performing the Fourier transform and finally doing the integral over the proper-time parameter. The expansion of $\tilde{G}$ in powers of B had been carried out previously in Ref. [15], where it was pointed out that the phase factor gives a non trivial contribution to the three-point amplitude

$$\begin{aligned}
e^{\Phi(y,x)}\cdot e^{\Phi(x,z)}\cdot e^{\Phi(z,y)} &= \exp\left(\frac{ie}{2}(z-y)^\alpha F_{\alpha\beta}(y-x)^\beta\right) \\
= e^{\Phi(x,z)}\cdot e^{\Phi(z,y)}\cdot e^{\Phi(y,x)} &= \exp\left(\frac{ie}{2}(y-x)^\alpha F_{\alpha\beta}(x-z)^\beta\right) \\
= e^{\Phi(z,y)}\cdot e^{\Phi(y,x)}\cdot e^{\Phi(x,z)} &= \exp\left(\frac{ie}{2}(x-z)^\alpha F_{\alpha\beta}(z-y)^\beta\right).
\end{aligned} \tag{10}$$

Any of the three forms can be used in the computation of the amplitude due to the trace over the Dirac indices. Since the phase is already linear in B, there are two kinds of contributions at first order: one when there are two factors $\tilde{G}_0$ and one $\tilde{G}_1$ and the other when there are three factors $\tilde{G}_0$ and the phase.

We start with the contribution without the phase which can be written directly in Fourier space (see Fig. 1)

$$M_1^{\rho\mu\nu}[p,p_1,p_2] = -(i)^3 e^3 T \sum_n \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}[\gamma^\mu \tilde{G}(k+p_1)\gamma^\rho \tilde{G}(k-p_2)\gamma^\nu \tilde{G}(k)] \\ +(\mu, p_1 \leftrightarrow \nu, p_2). \tag{11}$$

To proceed we choose the momentum of the incident photon along the OZ axis: $p^\mu = (\omega, 0, 0, p = \omega)$ and the magnetic field in the plane OYZ: $\mathbf{B} = (0, B_y, B_z)$ without loss of generality. Thus, in the collinear aproximation the momentum of the outgoing photons are $p_1^\mu = (\omega_1, 0, 0, p_1 = \omega_1)$ and $p_2^\mu = (\omega_2, 0, 0, p_2 = \omega_2)$. The polarization states are chosen following the convention of Adler [1]: $\epsilon_\parallel^\mu = (0,1,0,0)$, $\epsilon_\perp^\mu = (0,0,1,0)$.

Introducing the expression for $\tilde{G}(p)$ in Eq. (11) and taking the linear part in B we can compute the amplitude for the different channels. We have written a Mathematica code to carry out the Matsubara sums and the angular integrals. The Matsubara sums lead to derivatives of the Fermi distribution function which are integrated by parts, resulting at the end in the single integral of Eq. (3) defining the transverse mass. We obtain

$$M_1^{\parallel\to\perp\perp} = -\frac{1}{2}\frac{eB_y}{m^2}\frac{e}{\omega}\, m_t^2, \tag{12}$$

$$M_1^{\perp\to\parallel\perp} = \frac{1}{2}\frac{eB_y}{m^2}\frac{e}{\omega_1}\, m_t^2, \tag{13}$$

$$M_1^{\perp\to\perp\parallel} = \frac{1}{2}\frac{eB_y}{m^2}\frac{e}{\omega_2}\, m_t^2, \tag{14}$$

$$M_1^{\parallel\to\parallel\parallel} = \frac{1}{2}\frac{eB_y}{m^2}\frac{e(\omega_1^2+\omega_1\omega_2+\omega_2^2)}{\omega\omega_1\omega_2}\, m_t^2, \tag{15}$$

$$M_1^{\parallel\to\parallel\perp} = M_1^{\parallel\to\perp\parallel} = M_1^{\perp\to\parallel\parallel} = M_1^{\perp\to\perp\perp} = 0. \tag{16}$$

Now, in order to compute the contribution of the phase, we write the three-point function with the external legs in position space

$$\Gamma^{\rho\mu\nu}[x,y,z] = -(i)^3 e^3 \int d^4x' d^4y' d^4z' \mathrm{Tr}[\gamma^{\mu'}\tilde{G}_0(y'-x')\gamma^{\rho'}\tilde{G}_0(x'-z')\gamma^{\nu'}\tilde{G}_0(z'-y')] \tag{17}$$
$$\times D_{\rho,\rho'}(x-x')D_{\mu,\mu'}(y-y')D_{\nu,\nu'}(z-z')\left(\frac{ie}{2}(z'-y')^\alpha F_{\alpha\beta}(y'-x')^\beta\right) + (\mu, y \leftrightarrow \nu, z),$$

where $D_{\alpha,\alpha'}(x-x')$ are the propagators of the photons. In Fourier space this is given by

$$\Gamma^{\rho\mu\nu}[x,y,z] = \int \frac{d^4p}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} D_{\rho,\rho'}(p)D_{\mu,\mu'}(p_1)D_{\nu,\nu'}(p_2)e^{ipx}e^{-ip_1y}e^{-ip_2z} \\ \times(2\pi)^4\delta^4(p-p_1-p_2)M^{\rho'\mu'\nu'}(p,p_1,p_2). \tag{18}$$

Thus, the corresponding three-point amplitude reads

$$M_2^{\rho\mu\nu}[p,p_1,p_2] = -(i)^3 e^3 T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{ie}{2} F_{\alpha\beta} \frac{\partial}{\partial p_1^\alpha} \frac{\partial}{\partial p_2^\beta} \mathrm{Tr}\big[\big(\gamma^\mu \tilde{G}_0(k+p_1)\gamma^\rho \tilde{G}_0(k-p_2)\gamma^\nu \tilde{G}_0(k)\big)\big] + (\mu, p_1 \leftrightarrow \nu, p_2). \tag{19}$$

Proceeding as we did before for the other contribution we obtain an identical result. Thus the total amplitudes are

$$M_{\|\to\perp\perp} = -\frac{eB_y}{m^2}\frac{e}{\omega}\, m_t^2, \tag{20}$$

$$M_{\perp\to\|\perp} = \frac{eB_y}{m^2}\frac{e}{\omega_1}\, m_t^2, \tag{21}$$

$$M_{\perp\to\perp\|} = \frac{eB_y}{m^2}\frac{e}{\omega_2}\, m_t^2, \tag{22}$$

$$M_{\|\to\|\,\|} = \frac{eB_y}{m^2}\frac{e(\omega_1^2+\omega_1\omega_2+\omega_2^2)}{\omega\omega_1\omega_2}\, m_t^2, \tag{23}$$

$$M_{\|\to\|\perp} = M_{\|\to\perp\|} = M_{\perp\to\|\,\|} = M_{\perp\to\perp\perp} = 0. \tag{24}$$

The amplitudes with an odd number of perpendicularly polarized photons vanish. These results can be written in a compact form

$$M^{ijk}(p,p_1,p_2) = e^2 \frac{m_t^2}{m^2}\left[\frac{1}{\omega_2}\,\delta^{ij}\,V^k - \frac{1}{\omega}\,\delta^{jk}\,V^i + \frac{1}{\omega_1}\,\delta^{ki}\,V^j\right], \tag{25}$$

where $V^j = \epsilon^{jkm} B_k \hat{p}_m$ and i,j,k can take the values $1,2$ for polarizations $\|, \perp$ respectively.

Now we turn to the contribution at order B^2 coming from the pentagon graph. The computation proceeds along the lines of the previous one. The contribution at second order in B with no phases is computed as before. For the term linear in the phase we start from Eq.(19), but now the factor inside the trace must be replaced by its first order expression in B. Finally, for the second order contribution of the phase we have an expression similar to Eq. (19)

$$M^{\rho\mu\nu}[p,p_1,p_2] = -\frac{1}{2}(i)^3 e^3 T \sum_n \int \frac{d^3k}{(2\pi)^3}\left(\frac{ie}{2}\right)^2 F_{\alpha\beta}F_{\sigma\eta} \frac{\partial}{\partial p_1^\alpha}\frac{\partial}{\partial p_2^\beta}\frac{\partial}{\partial p_1^\sigma}\frac{\partial}{\partial p_2^\eta} \mathrm{Tr}\big[\big(\gamma^\mu \tilde{G}_0(k+p_1)\gamma^\rho \tilde{G}_0(k-p_2)\gamma^\nu \tilde{G}_0(k)\big)\big] + (\mu, p_1 \leftrightarrow \nu, p_2). \tag{26}$$

The results of each part alone are not very illustrative, so we give only the total amplitude

$$M_{\parallel\to\parallel\,\perp} = -ie^3 \left(\frac{B_y B_z}{B_{cr}^2}\right)\left(\frac{\omega_1\omega + 3\omega_2^2}{\omega\omega_1\omega_2^2}\right) n_e, \tag{27}$$

$$M_{\parallel\to\perp\parallel} = -ie^3 \left(\frac{B_y B_z}{B_{cr}^2}\right)\left(\frac{\omega_2\omega + 3\omega_1^2}{\omega\omega_1^2\omega_2}\right) n_e, \tag{28}$$

$$M_{\perp\to\parallel\,\parallel} = ie^3 \left(\frac{B_y B_z}{B_{cr}^2}\right)\left(\frac{3\omega_1^2 + 5\omega_1\omega_2 + 3\omega_2^2}{\omega^2\omega_1\omega_2}\right) n_e, \tag{29}$$

$$M_{\perp\to\perp\perp} = -ie^3 \left(\frac{B_y B_z}{B_{cr}^2}\right)\left(\frac{\omega_1^2 + \omega_1\omega_2 + \omega_2^2}{\omega\omega_1\omega_2}\right)^2 n_e, \tag{30}$$

$$M_{\parallel\to\perp\perp} = M_{\perp\to\parallel\,\perp} = M_{\perp\to\perp\parallel} = M_{\parallel\to\parallel\,\parallel} = 0, \tag{31}$$

with

$$n_e = 2\int \frac{d^3p}{(2\pi)^3}(n_F(\varepsilon_p - \mu) - n_F(\varepsilon_p + \mu)) \tag{32}$$

being the net electron density. The amplitudes for an even number of perpendicularly polarized photons vanish. These results can also be written in a compact form

$$M^{ijk}(p, p_1, p_2) = -ie^3 \left(\frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{B_{cr}^2}\right) n_e \tag{33}$$
$$\times \left[\left(\frac{1}{\omega^2} - \frac{3}{\omega_1\omega_2}\right)\delta^{jk}\, B_\perp^i + \left(\frac{1}{\omega_1^2} + \frac{3}{\omega\omega_2}\right)\delta^{ik}\, B_\perp^j + \left(\frac{1}{\omega_2^2} + \frac{3}{\omega\omega_1}\right)\delta^{ij}\, B_\perp^k\right],$$

where $B_\perp^i = B^i - (\mathbf{B}\cdot\hat{\mathbf{p}})\hat{p}^i$ and i, j, k can take the values $1, 2$ for $\parallel, \perp$ respectively.

An important issue concerning the outlined computations is its consistency with gauge invariance. Following the same procedure, we have computed other components of the three-point amplitude and we have explicitely verified that the Ward identities in the collinear approximation for null vectors, $M^{0\mu\nu} - M^{3\mu\nu} = 0$, are satisfied.

III. THE ABSORPTION COEFFICIENT

In order to obtain the absorption coefficient a phase space integration with the proper measure has to be performed. When the effects of the magnetic field are considered in the dispersion relations, Adler [1] showed by using arguments based on CP invariance and kinematics, that in the vacuum the only allowed channel is $\parallel\to\perp\perp$. As we have mentioned above, for high energy photons ($\omega \gg \omega_0$) propagating in a magnetized plasma, the influence of density or temperature on the dispersion relation is small and we can expect that the same channel is the only one allowed for contributions which are even in the chemical potential. For this decay we have

$$d\kappa_{\|\to\perp\perp} = \frac{1}{2}\frac{1}{2\omega 2\omega_1 2\omega_2}\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,(2\pi)^4\delta^4(k-k_1-k_2)\times|M_{\|\to\perp\perp}|^2, \tag{34}$$

where the enhancement Bose factors appearing in the probability have been replaced by one, because we consider the case $\omega_1, \omega_2 \gg T$. If dispersion due to the magnetic field is also neglected, the integrals reduce to $\int_{\omega_0}^{\omega-\omega_0} dk_1$, which can be approximated by the overall integral, so that

$$\kappa_{\|\to\perp\perp} = \frac{|M_{\|\to\perp\perp}|^2}{32\pi\,\omega} = \frac{\alpha}{8}\left(\frac{B\sin\theta}{B_{cr}}\right)^2\left(\frac{m}{\omega}\right)^3\left(\frac{m_t}{m}\right)^4 m. \tag{35}$$

For the pentagon contribution the kinematical constraints are the same as before, but the CP arguments now allow only processess with an odd number of perpendicularly polarized photons. When both constraints are taken into account the only allowed channel is $\|\to\|_1\perp_2$ or $\|\to\perp_1\|_2$. The absorption coefficient is given by an expression similar to Eq. (34) with $|M_{\|\to\perp\perp}|^2$ replaced by $|M_{\|\to\|\perp}|^2+|M_{\|\to\perp\|}|^2$. Now, the integrand depends on ω_1 and ω_2 and hence the final result displays an explicit dependence on the cutoff

$$\kappa_{\|\to\|\perp} = \frac{\pi^2\alpha^3}{3}\left(\frac{B}{B_{cr}}\right)^4\sin^2(2\theta)\left(\frac{m}{\omega}\right)^2\left(\frac{m}{\omega_0}\right)^3\left(\frac{n_e}{m^3}\right)^2 m. \tag{36}$$

We stress again that an application of the above formulae for low energy photons is misleading, as the formulae themselves show. At low energy, the propagating modes in the plasma are longitudinal or transverse plasmons whose dispersion relations differ significantly from that we have assumed, $\omega = p$.

We show the explicit dependence on the plasma parameters for two opposite regimes. For a completely degenerated electron plasma we have

$$\kappa_{\|\to\perp\perp} = \frac{\alpha^3}{2\pi^2}\left(\frac{B\sin\theta}{B_{cr}}\right)^2\left(\frac{m}{\omega}\right)^3\times\left[\ln\left(\frac{\mu+\sqrt{\mu^2-m^2}}{m}\right)-\frac{\mu}{m}\sqrt{\frac{\mu^2-m^2}{m^2}}\right]^2 m, \tag{37}$$

$$\kappa_{\|\to\|\perp} = \frac{\alpha^3}{27\pi^2}\left(\frac{B}{B_{cr}}\right)^4\sin^2(2\theta)\left(\frac{m}{\omega}\right)^2\left(\frac{m}{\omega_0}\right)^3\left(\frac{\mu^2-m^2}{m^2}\right)^3 m, \tag{38}$$

and for high temperature $T \gg m$ and $\mu = 0$, we obtain at order B^2

$$\kappa_{\parallel \to \perp\perp} = \frac{\pi^2 \alpha^3}{18} \left(\frac{B \sin\theta}{B_{cr}} \right)^2 \left(\frac{T}{\omega} \right)^3 T. \tag{39}$$

Obviously, there is no contribution at order B^4.

In order to find an estimate of matter effects in photon splitting, we compute the absorption coefficient for two astrophysical objects. For a white dwarf we consider the following values of the parameters [16]

$$\frac{B}{B_{cr}} \sim 2 \cdot 10^{-6}, \quad \frac{T}{m} \sim 2 \cdot 10^{-3}, \quad \frac{\mu}{m} \sim 1.02. \tag{40}$$

The transverse mass and the charge density can be estimated with formulas (3, 32), giving $m_t \sim 5 \cdot 10^{-3}\, m,\ n_e \sim 2.7 \cdot 10^{-4} m^3$. In order to satisfy (5) we need $\omega \gg 2.5 \cdot 10^5\, m$. Taking $\theta = \pi/4$ and $\omega \sim 10^6\, m$ we get the following estimates

$$\kappa_{\parallel \to \perp\perp} \simeq 10^{-42}\, m, \quad \kappa_{\parallel \to \parallel\perp} \simeq 10^{-64}\, m. \tag{41}$$

The value in the vacuum from the hexagon is $\kappa_{\parallel \to \perp\perp} \simeq 3.5 \cdot 10^{-17}\, m$. Here, the effects of matter are completely irrelevant. For a neutron star we can take the values [16]

$$\frac{B}{B_{cr}} \sim 2 \cdot 10^{-1}, \quad \frac{T}{m} \sim 1, \quad \frac{\mu}{m} \sim 600. \tag{42}$$

Now we have $m_t \sim 40.9\, m$, $n_e \sim 7.3 \cdot 10^6\, m^3$ and $\omega_0 \sim 2 \cdot 10^4\, m$. Again, taking $\theta = \pi/4$ and $\omega \sim 10^5\, m$ we get the estimate

$$\kappa_{\parallel \to \perp\perp} \simeq 5 \cdot 10^{-14}\, m, \quad \kappa_{\parallel \to \parallel\perp} \simeq 1.4 \cdot 10^{-18}\, m, \tag{43}$$

while the value of the hexagon in the vacuum is $\kappa_{\parallel \to \perp\perp} \simeq 3.5 \cdot 10^8\, m$. We see that the effects of the medium are neglible by many orders of magnitude. In addition, the results for the neutron star do not change appreciably in the range of temperatures from $T/m = 10^{-3}$ to $T/m = 1$.

IV. SUMMARY

In this work we have computed the contribution of order B^2 and B^4 to the photon splitting process at finite temperature or density in the collinear aproximation. We have computed directly the three-photon amplitude using fermionic propagators in the presence of a magnetic external field for photons with $\omega = p$. The phase of the propagator contributes non trivially to the amplitude. In order to have photons with $\omega \simeq p$ we have restricted ourselves to photons with $\omega \gg \omega_0$ so the effects of the medium on the dispersion relations can be neglected compared to those of the magnetic field. This condition leads to consider very high energy photons, for which the effects of the medium are neglible compared with the perturbative vacuum contribution coming from the hexagon, as seen in explicit computations. It is not excluded that for lower energy photons, when the full effects of the dispersion relations must be taken into account, the box contribution is more significant.

A natural extension of this work is the study of the matter-induced three-point vertex at strong field, a regime with possible astrophysical applications. At strong field, the modification of the dispersion relations due to vacuum effects play a more significant role [5] than at weak fields.

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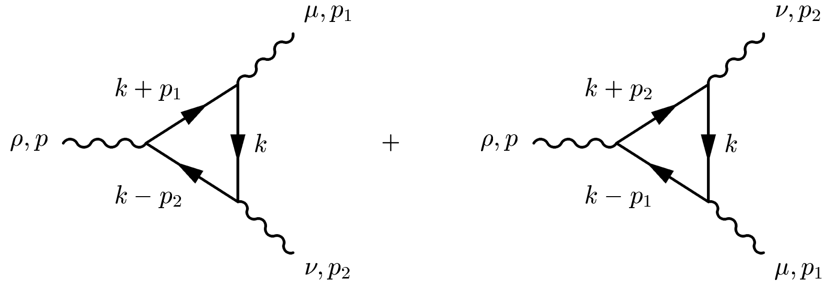

FIG. 1. The photon splitting amplitude. The solid lines represents fermionic propagators in the presence of a magnetic external field.